ANALYTICAL DESIGN OF CONTROLLERS IN STOCHASTIC SYSTEMS WITH VELOCITY-LIMITED CONTROLLING ACTION

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The selection of the control process for a system in which the controlled object is subject to random changes is discussed. The controlling action ξ is formed, subject to the condition that the integral quality estimate is a minimum. This estimate is a given function of the off-balance of coordinates x_i, the controlling action ξ and the rate of change $\dot{\xi}$ of ξ during the transient process. The problem is investigated using the Liapunov-function methods [1,2], modified in accordance with the principles of dynamic programming [3,4]. The present results generalize those reported in [5] to the case of stochastic systems.

1. Preliminary remarks. Consider a control system in which the transient process is described by the stochastic differential equations of perturbed motion

$$\frac{dx_i}{dt} = f_i[x_1, \ldots, x_n, \xi, \eta(t)] \qquad (i = 1, \ldots, n) \tag{1.1}$$

Here, x_i are the deviations of the true values of the coordinates of the controlled vector quantity from the prescribed (unperturbed) values $x_i = 0$ $(i = 1, \ldots, n)$, and ξ is the control action produced by the controller. The particular feature of the system is that it is subject to random changes during the control process. This is taken into account in Equation (1) by introducing the random variable $\eta(t)$.

We shall assume that f_i are known continuous functions of their arguments which satisfy the Lipschitz conditions in the domain G of the $\{x, \xi, \eta\}$ space, $f_i[0, \ldots, 0, \eta(t)] = 0$, and the variable $\eta(t)$

represents the Markov random process [6, p. 79]. Under these conditions and provided the control action ξ is known, Equations (1.1) describe the random transient process $x(t)$.

The control action ξ of the controller will be determined from the condition

$$J = \int_0^\infty M\{\omega[x_1(t), \ldots, x_n(t), \xi(t), \dot{\xi}(t)]\} dt = \min \qquad (1.2)$$

where the symbol M denotes the mathematical expectation of the random quantity ω which is a given non-negative function of its arguments. Since (1.2) includes $\dot{\xi}$, we shall seek the equation for the optimum controller in the form*

$$\dot{\xi} = \zeta[x_1, \ldots, x_n, \xi, \eta] \qquad (1.3)$$

The aim of the present paper is to investigate the form of the function ζ which would ensure the stability of the unperturbed motion $x_i = 0$ $(i = 1, \ldots, n)$ and would satisfy the condition (1.2).

2. Formulation of the problem. Let $P[Q/L]$ be the probability of an event Q subject to the condition L, $M\{\omega/L\}$ be the mathematical expectation of the random quantity ω subject to the condition L and let $o(\Delta t)$ be an infinitely small quantity of a higher order than Δt (the small quantity Δt is assumed to be positive throughout).

Let us now describe the statistical properties of the random function $\eta(t)$. Let us confine our attention to the case where there are two functions, namely $q(\alpha)$ and $q(\alpha, \beta)$, which describe the time changes of $\eta(t)$ and are defined by [6, pp. 231-245]

$$P[\eta(t+\Delta t) = \alpha / \eta(t) = \alpha] = 1 - q(\alpha)\Delta t + o(\Delta t) \qquad (2.1)$$

$$P[\eta(t+\Delta t) = \beta, \eta(t+\Delta t) \neq \alpha / \eta(t) = \alpha] = q(\alpha, \beta)\Delta t + o(\Delta t) \qquad (2.2)$$

$$(\alpha, \beta = 1, \ldots, m+1)$$

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* If the quantity $\dot{\xi}$ does not enter into the right-hand side of (1.2), then the equation for ξ can be sought in the form of the ideal controller $\xi = \xi[x_1, \ldots, x_n, \eta]$. Among the arguments of the function ζ in (1.3) there is the function η. This means that it is assumed that $\eta(t)$ can be measured and the corresponding signal can be fed into the controller.

i.e. the probability that $\eta(t)$ will remain constant ($\eta(t) = \alpha$) over the small interval Δt is approximately equal to $1 - q(\alpha)\Delta t$ and the probability of the single change $\eta(t) = \alpha \to \eta(t + \Delta t) = \beta$ in this interval is approximately equal to $q(\alpha, \beta)\Delta t$.

In accordance with [6, p. 242], we shall assume that the realizations of the random process $\eta(t)$ are the step functions $\eta^{(p)}(t)$ (except for a set of zero probability realizations which will be neglected).

If the function ζ in (1.3) has been chosen (and is continuous and satisfies, for example, the Lipschitz conditions with respect to x_i, ξ, η), then for each realization $\eta^{(p)}(t)$ and for each initial condition $\{x_{i0}, \xi_0, t = t_0\}$ Equations (1.1) and (1.3) define the continuous realization $x^{(p)}(x_0, \xi_0, t_0, t, \eta^{(p)})$, $\xi^{(p)}(x_0, \xi_0, t_0, t, \eta^{(p)}$ of the random solution $x(t)$, $\xi(t)$ describing the transient process in the system. If this random process is considered in the $\{x_1, \ldots, x_n, \xi, \eta\}$ space, then it turns out to be a Markov random process. Let us denote by $\{x(t), \xi(t), \eta(t)\}/x_0, \xi_0, \eta_0, t_0$ the Markov random vector function, generated by the initial conditions $x_i = x_{i0}$, $\xi = \xi_0$, $\eta = \eta_0$ when $t = t_0$ and which for $t \geqslant t_0$ is a solution of (1.1) and (1.3), as defined above. Unless the opposite is stated, it will be assumed that the functions f_i and ζ are defined in the entire $\{x, \xi\}$ space (the domain G is defined by $\{-\infty < x_i < \infty \ (i = 1, \ldots, n), -\infty < \xi < \infty, \eta_1 \leqslant \eta \leqslant \eta_2\}$).

Expression (1.2) can now be written out in the more detailed form

$$J_\zeta[x_0, \xi_0, \eta_0] = \int_0^\infty \boldsymbol{M}\{\omega[x(t), \xi(t), \dot{\xi}(t)]/x_0, \xi_0, \eta_0, t_0 = 0\}\, dt \qquad (2.3)$$

and in accordance with our assumptions it will be a function of the initial conditions x_0, ξ_0, η_0, $t_0 = 0$ (when the function ζ in (1.3) is fixed). The quantity on the left-hand side of (2.3) will also be looked upon as a functional J_ζ of ζ, since the function ζ defines the solutions $\{x(t), \xi(t), \eta(t)\}$ and hence the value of the integral in (2.3) also.

The problem consists of the determination of the function $\zeta^\circ[x, \xi, \eta]$, which obeys the following conditions.

Condition 2.1. The unperturbed motion $x = 0$, $\xi = 0$ when $\zeta = \zeta^\circ$ in Equation (1.3) should be asymptotically stable* (definition 1.2, [7],

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* For the sake of completeness let us re-state these definitions.

Definition 2.1. The solution $x = 0$, $\xi = 0$ of (1.1) and (1.3) (unperturbed motion) will be defined as stable in probability if, for any two numbers, $\epsilon > 0$, $q > 0$, which are as small as desired, one can

p. 810) with respect to any initial perturbations x_{i0}, ξ_0.

Condition 2.2. The integral (2.3) should be finite for any initial conditions and for every initial condition x_{i0}, ξ_0, η_0

$$J_{\zeta^\circ}[x_0, \xi_0, \eta_0] = \min_{\zeta} J_{\zeta}[x_0, \xi_0, \eta_0] \qquad (2.4)$$

for functions ζ from a defined class $\{\zeta\}$ of these functions. Below, this class will be defined as the set of continuous functions ζ, $\zeta[0, 0, \eta] = 0$.

The problem under consideration is one of the statistical optimum-control problems (see [3, 8, 9]). It is a generalized form of the problem investigated in [5].

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find a number $\delta > 0$ which is such that for any solution of (1.1) and (1.3) which at $t = t_0$ satisfies the inequality $x_{10}^2 + \ldots + x_{n0}^2 + \xi_0^2 \leqslant \delta^2$ for $t > t_0$, will satisfy the condition

$$P[\varepsilon, t/\delta, t_0] > 1 - q$$

Here, $P[\epsilon, t/\delta, t_0]$ is the probability that for $t > t_0$ either

$$x_1^2(t) + \ldots + x_n^2(t) + \xi^2(t) < \varepsilon^2$$

or

$$x_1^2(\tau) + \ldots + x_n^2(\tau) + \xi^2(\tau) < \varepsilon^2 \qquad \text{for } t_0 \leqslant \tau < t$$

which satisfies if one assumes a cut-off in the realization $x^{(p)}(t)$, $\xi^{(p)}(t)$ on leaving the ϵ-neighborhood of the point $x = 0$, $\xi = 0$.

Definition 2.2. The perturbed motion $x = 0$, $\xi = 0$ will be defined as asymptotically stable in probability if it is stable in the sense of the definition (2.1) and, moreover, the following condition is satisfied for any $\epsilon > 0$ $q > 0$:

$$P[x_1^2(t) + \ldots + x_n^2(t) + \xi^2(t) < \varepsilon^2 \text{ for } t \geqslant t_0 + T[H_0, \varepsilon, q] / \\ / x_{10}^2 + \ldots + x_{n0}^2 + \xi_0^2 \leqslant H_0] > 1 - q$$

Here, H_0 is a constant which limits the initial conditions.

In the present paper we shall confine ourselves to the case where the domain of allowed deviations x_{i0}, ξ_0 covers the entire $\{x, \xi\}$ space (the constant H_0 can be any positive number).

3. Method of solution. We shall describe the method of solution based on the Liapunov-function method. The application of this method to statistical stability problems is described in [7].

In the present section the problem (2.4) is considered in the light of ideas reported in [9] and [10] in connection with optimum speed of response. As in those papers, we shall use ideas from the theory of dynamic programming [2,3,5]. We shall assume that the final functions $v^{\circ}(x_1, \ldots, x_n, \xi, \eta)$, satisfy the following conditions.

Condition 3.1. The function $v^{\circ}(x, \xi, \eta)$ is a positive-definite function ([1, p. 80] and [7, p. 811]) for all values x, ξ and the possible values of η. The function $v^{\circ}(x, \xi, \eta)$ can have an infinitely large low limit [11, p. 36], i.e.

$$v^{\circ}(x, \xi, \eta) \geqslant w(x, \xi) > 0 \quad \text{for } \{x, \xi\} \neq 0,$$
$$w(x, \xi) \to \infty \quad \text{for } \{x, \xi\} \to \infty$$

Condition 3.2. The derivative $(dM\{v^{\circ}\}/dt)_{\zeta^{\circ}}$ [9,10] is, in view of (1.1) and (1.3), a negative-definite function at $\zeta = \zeta^{\circ}$ and is equal to $-\omega[x, \xi, \dot{\xi}]$, i.e.*

$$\left(\frac{dM\{v^{\circ}\}}{dt}\right) = -\omega[x_1, \ldots, x_n, \xi, \zeta^{\circ}] \tag{3.1}$$

Condition 3.3. The quantity $dM\{v^{\circ}\}/dt + \omega[x, \xi, \zeta^{\circ}]$ reaches a minimum when $\zeta = \zeta^{\circ}$, i.e.

$$\left(\frac{dM\{v^{\circ}\}}{dt}\right)_{\zeta^{\circ}} + \omega[x_1, \ldots, x_n, \xi, \zeta^{\circ}] = \min_{\zeta}\left[\left(\frac{dM\{v^{\circ}\}}{dt}\right)_{\zeta} + \omega[x_1, \ldots, x_n, \xi, \zeta]\right] \tag{3.2}$$

for ζ belonging to the allowed class of functions $\{\zeta\}$.

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* In the terminology of stochastic processes $dM\{v^{\circ}\}/dt$ is determined by an infinitely small differential operator for the process [12]. This quantity is obtained as follows: the point $x(t) = x$, $\xi(t) = \xi$, $\eta(t) = \eta$ generates for $r > t$ the set of random realizations $\{x(r), \xi(r), \eta(r)\}/x, \xi, \eta, t$; the symbol $dM\{v^{\circ}\}/dt$ is defined in the usual way as the limit of the ratio $\Delta M\{v^{\circ}\}/\Delta t$ when $\Delta t \to 0$, where

$$\Delta M\{v^{\circ}\} = M\{v^{\circ}(x(t+\Delta t), \xi(t+\Delta t), \eta(t+\Delta t))/x(t), \xi(t), \eta(t), t\} - v^{\circ}(x(t), \xi(t), \eta(t))$$

Condition 3.4. The function $v^\circ(x, \xi, \eta)$ can have an infinitely small upper limit [1, p. 81] and when $\omega[x, \xi, \eta] \to 0$.

The function v° which satisfies conditions 3.1 to 3.4 will be called an optimum Liapunov function for the problem defined by (2.4). The function ζ°, which satisfies 3.2 and 3.3, determines the optimum control law*, i.e. when $\zeta = \zeta^\circ$ in (1.3) the solutions $\{x(t), \xi(t), \eta(t)\}$ satisfy conditions 2.1 and 2.2. In fact, the asymptotic stability of the solution $x = 0$, $\xi = 0$ for the probability (2.1) subject to 3.1, 3.2 and 3.4 follows from the theorems given in [7] (pp. 812-820). These theorems were established in [7] for the random function $\eta(t)$ of a more special type, although the discussion applies for variable $\eta(t)$ of the more general form described above. The appropriate proof is analogous to that quoted in [7] and will not be repeated here. We shall merely note that it follows from this proof that in addition to the asymptotic stability, as defined by definition 1.2 [7], there is also the probabilistic stability of the solution $x = 0$, $\xi = 0$ in the following sense: for any numbers $\epsilon > 0$, $q > 0$ one can find $\delta > 0$ such that the following inequality is obeyed:

$$P[x_1^2(t) + \ldots + x_n^2(t) + \xi^2(t) < \varepsilon^2 \ \text{for} \ t \geqslant t_0 / $$
$$/ x_{10}^2 + \ldots + x_{n_0}^2 + \xi_0^2 \leqslant \delta^2] > 1 - q.$$

We shall now sketch the proof that the requirement 3.2 follows from conditions 3.3 and 3.4.

To begin with, we have the following equation from (3.1) after averaging $\{x(t), \xi(t), \eta(t)\}/x_0, \xi_0, \eta_0, t_0 = 0$ over random quantities

$$\left(\frac{dM\{v^\circ(x(t), \xi(t), \eta(t))\}}{dt}\right)_{\zeta^\circ} = M\left\{\left(\frac{dM\{v^\circ\}}{dt}\right)_{\zeta^\circ}\right\} = -M\{\omega[x(t)\,\xi(t), \zeta^\circ(t)]\} \quad (3.3)$$

It follows from (3.3) that $M\{v^\circ(x(t), \xi(t), \eta(t))\}$ decreases with t. Integrating (3.3) with respect to t between $t_0 = 0$ and $t = T$, we have

$$M\{v^\circ(x(T), \xi(T), \eta(T))\}_{\zeta^\circ} - v^\circ(x_0, \xi_0, \eta_0)$$
$$= -\int_0^T M\{\omega[x(t), \xi(t), \dot{\xi} = \zeta^\circ(t)]\}_{\zeta^\circ}\, dt \quad (3.4)$$

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* It is assumed that the functions v° and ξ° are sufficiently smooth so that one can speak of the existence of solutions of (1.1) and (1.3) and use the derivative $dM\{v^\circ\}/dt$ and the transformations (3.3) to (3.9).

We conclude from (3.4) that the integral on the right-hand side of this equation converges for $T \to \infty$. Since ω is not negative, this means that $\lim M\{\omega\} = 0$ when $t \to \infty$, i.e. according to condition 3.4 the monotonic function $M\{v^\circ(x(t), \xi(t), \eta(t))\}$ will vanish in the limit as $T \to \infty$*.

It therefore follows from (3.4) that

$$v^\circ(x_0, \xi_0, \eta_0) = \int_0^\infty M\{\omega[x(t), \xi(t), \dot{\xi}(t)]\}_{\zeta^\circ} dt \tag{3.5}$$

i.e. $J_{\zeta^\circ}[x_0, \xi_0, \eta_0]$ remains finite and $J_{\zeta^\circ} = v^\circ$. Let us now assume that there exists a function $\zeta^*[x, \xi, \eta] \neq \zeta^\circ[x, \xi, \eta]$ which is such that when $\zeta = \zeta^*$ in (1.3) the solutions $\{x(t), \xi(t), \eta(t)\}_{\zeta^*}$ of (1.1) and (1.3) yield the following inequality for a certain initial condition x_0, ξ_0, η_0 (when $t_0 = 0$):

$$J_{\zeta^*}[x_0, \xi_0, \eta_0] < J_{\zeta^\circ}[x_0, \xi_0, \eta_0] \tag{3.6}$$

It follows from (3.2) that

$$\left(\frac{dM\{v^\circ\}}{dt}\right)_{\zeta^*} \geqslant -\omega[x, \xi, \dot{\xi} = \zeta^*] \tag{3.7}$$

Averaging (3.7) over the random quantities, we have

$$\{x(t), \xi(t), \eta(t)\}_{\zeta^*} / x_0, \xi_0, \eta_0, t_0 = 0$$

and integrating with respect to t as before we are led to the inequality

$$\begin{aligned} M\{v^\circ(x(T), \xi(T), \eta(T))\}_{\zeta^*} - v^\circ(x_0, \xi_0, \eta_0) \geqslant \\ \geqslant -\int_0^T M\{\omega[x(t), \xi(t), \zeta^*(t)]\}_{\zeta^*} dt \end{aligned} \tag{3.8}$$

It follows from (3.6) that when $T \to \infty$, the integral on the right-hand side of (3.8) converges, i.e.

$$v^\circ(x_0, \xi_0, \eta_0) \leqslant \int_0^\infty M\{\omega[x(t), \xi(t), \zeta^*(t)]\}_{\zeta^*} dt = J_{\zeta^*}[x_0, \xi_0, \eta_0] \tag{3.9}$$

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* We confine our attention to the case where from $M\{\omega\} \to 0$ one can conclude that $M\{v^\circ\} \to 0$.

In fact, it follows from the convergence of the integral on the right-hand side of (3.8) that $\lim M\{\omega^\circ\}_{\zeta*} = 0$ when $t \to \infty$, i.e. according to 3.4 (see footnote on p.633) $\lim M\{v^\circ\}_{\zeta*} = 0$ $(t \to \infty)$. The inequality (3.9) contradicts (3.5) and our assumption (3.6). This contradiction confirms the optimum property of ζ°.

The problem is therefore reduced to the determination of the functions $v^\circ(x, \xi, \eta)$ and $\zeta^\circ[x, \xi, \eta]$.

4. Equations for the optimum functions v°, ζ°. The equations for v° and ζ° can be verified from the conditions (3.1) and (3.2). It is, however, necessary to know the expression for $dM\{v^\circ\}/dt$ in terms of the right-hand sides of Equations (1.1) and (1.3) and the statistical characteristics $q(\alpha)$ and $q(\alpha, \beta)$ given by (2.1) and (2.2). Let us now derive this expression. For the sake of simplicity, we shall assume that $\eta(t)$ is such that $\eta_1 \leqslant \eta \leqslant \eta_2$, $\eta_1 > -\infty$, $\eta_2 < \infty$.

In order to compute the derivative $dM\{v^\circ\}/dt$ from the limit

$$\frac{dM\{v^\circ\}}{dt} = \lim \frac{\Delta M\{v^\circ\}}{\Delta t} \quad \text{for } \Delta t \to 0 \tag{4.1}$$

one can, in calculating $\Delta M\{v^\circ\}$, neglect the terms of the order of $o(\Delta t)$; then one can proceed as follows. Suppose that the following values were realized at the time t, $x(t) = x$, $\xi(t) = \xi$, $\eta(t) = \eta$, $v^\circ(t) = v^\circ(x(t), \xi(t), \eta(t))$. Then, during the time interval Δt which is such that $(t \leqslant r \leqslant t + \Delta t)$, the following mutually exclusive events can occur [13].

Event A. The quantity $\eta(r)$, $t \leqslant r \leqslant t + \Delta t$ remains constant, i.e. $\eta(t) = \eta(r) = \eta(t + \Delta t) = \text{const} = \eta$. According to (2.1), the corresponding probability is

$$P(A) \approx 1 - q(\eta)\Delta t$$

Event B. The quantity $\eta(r)$ when $t \leqslant r \leqslant t + \Delta t$ changes in value once. According to (2.2), the corresponding probability is

$$P(B) \approx q(\eta)\Delta t$$

Event B can in turn be split into mutually exclusive events B_k [13, p. 424], in which the quantity $\eta(r)$ changes its value only once and so that $\eta(t + \Delta t) = \beta_k$, where $\beta_k (k = 1, \ldots, m)$ is a set of values of $\eta(r)$ which are different from η. According to (2.2) the corresponding probability is

$$P(B_k) \approx q(\eta, \beta_k)\Delta t$$

In the case of event A, Equations (1.1) and (1.3) will behave over the interval Δt as ordinary differential equations and the increment of the function v°, i.e. $\Delta_A v^\circ$ will in this case be calculable in accordance with the usual rules [1, pp. 80-90]

$$\Delta_A v^\circ \approx \sum_{i=1}^{n} \frac{\partial v^\circ(x, \xi, \eta)}{\partial x_i} \Delta x_i + \frac{\partial v^\circ(x, \xi, \eta)}{\partial \xi} \Delta \xi \approx$$

$$\approx \left[\sum_{i=1}^{n} \frac{\partial v^\circ(x, \xi, \eta)}{\partial x_i} f_i[x, \xi, \eta] + \frac{\partial v^\circ(x, \xi, \eta)}{\partial \xi} \zeta[x, \xi, \eta] \right] \Delta t \tag{4.2}$$

In the case of a B_k event, it is clear that*

$$\Delta_{Bk} v^\circ = v^\circ(x, \xi, \beta) - v^\circ(x, \xi, \eta) + O(\Delta t) \tag{4.3}$$

Bearing in mind (4.2) and (4.3), we obtain the following formula for the mathematical expectation $\Delta M\{v^\circ\}$ (to within terms of the order of $O(\Delta t)$):

$$\Delta M\{v^\circ\} \approx P(A)\,\Delta_A v^\circ + \sum_{k=1}^{m} P(B_k)\,\Delta_{B_k} v^\circ$$

$$= \Delta t \left[\sum_{i=1}^{n} \frac{\partial v^\circ(x, \xi, \eta)}{\partial x_i} f_i[x, \xi, \eta] + \frac{\partial v^\circ(x, \xi, \eta)}{\partial \xi} \zeta[x, \xi, \eta] + \right.$$

$$\left. + \sum_{k=1}^{m} q(\eta, \beta_k)[v^\circ(x, \xi, \beta_k) - v^\circ(x, \xi, \eta)] \right] \tag{4.4}$$

since in accordance with (2.1) and (2.2)

$$\sum_{k=1}^{m} q(\eta, \beta_k) = q(\eta)$$

we have

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* The symbol $O(\Delta t)$ denotes an infinitely small quantity whose order of magnitude is not less than Δt, i.e. at any rate $O(\Delta t) \to 0$ when $\Delta t \to 0$.

$$\Delta M\{v^\circ\} \approx \Delta t\left[\sum_{i=1}^{n} \frac{\partial v^\circ(x, \xi, \eta)}{\partial x_i} f_i[x, \xi, \eta] + \frac{\partial v^\circ(x, \xi, \eta)}{\partial \xi} \times \right.$$

$$\left. \times \zeta[x, \xi, \eta] + \sum_{k=1}^{m} q(\eta, \beta_k) v^\circ(x, \xi, \beta_k) - q(\eta) v^\circ(x, \xi, \eta)\right] \qquad (4.5)$$

Dividing (4.5) by Δt and passing on to the limit as given by (4.1), we have

$$\frac{dM\{v^\circ\}}{dt} = \sum_{i=1}^{n} \frac{\partial v^\circ(x, \xi, \eta)}{\partial x_i} f_i[x, \xi, \eta] + \frac{\partial v^\circ(x, \xi, \eta)}{\partial \xi} \zeta[x, \xi, \eta] +$$

$$+ \sum_{k=1}^{m} v^\circ(x, \xi, \beta_k) q(\eta, \beta_k) - q(\eta) v^\circ(x, \xi, \eta) \qquad (4.6)$$

We can now write down the equations for v° and ζ°. As a consequence of (3.1) and (4.6) the first equation is of the form

$$\sum_{i=1}^{n} \frac{\partial v^\circ(x, \xi, \eta)}{\partial x_i} f_i[x, \xi, \eta] + \frac{\partial v^\circ(x, \xi, \eta)}{\partial \xi} \zeta^\circ[x, \xi, \eta] +$$

$$+ \sum_{k=1}^{m} v^\circ(x, \xi, \beta_k) q(\eta, \beta_k) - q(\eta) v^\circ(x, \xi, \eta) + \omega[x, \xi, \zeta^\circ] = 0 \qquad (4.7)$$

The second equation can be obtained from (4.7) by differentiation with respect to ζ, since, in accordance with the above (p. 631), when $\zeta = \zeta^\circ$, the left-hand side of (4.7) is a minimum. The second equation is

$$\frac{\partial v^\circ(x, \xi, \eta)}{\partial \xi} + \frac{\partial \omega[x, \xi, \zeta^\circ]}{\partial \zeta} = 0 \qquad (4.8)$$

If, in addition, ζ° is subject to a supplementary condition, e.g. $|\zeta^\circ| \leqslant 1$, then the minimum of the left-hand side of (4.7) must be sought subject to this condition.

Thus the problem is reduced to the solution of (4.7) and (4.8). The latter are partial differential equations and their general solution is very difficult. However, an approximate method of solution of these equations is possible.

5. Approximate method of solution of the equations for v° and ζ°. We shall describe a method for solving the optimum problem defined above which involves the introduction of a parameter ϑ . This method was outlined for the deterministic case in [10]. Let us consider instead of the system (1.1), (1.3), the auxiliary set of equations

$$\frac{dx_i}{dt} = \varphi_i [x, \xi, \eta; \vartheta] \qquad (i = 1, \ldots, n), \qquad \frac{d\xi}{dt} = \zeta [x, \xi, \eta; \vartheta] \quad (5.1)$$

and let us solve the problem of determination of the stabilizing function ζ°, which minimizes the functional

$$J_\zeta [x_0, \xi_0, \eta_0; \vartheta] = \int_0^\infty M \{\psi [x(t), \xi(t), \dot{\xi}(t); \vartheta] / x_0, \xi_0, \eta_0, t_0 = 0\} \, dt \quad (5.2)$$

for this system of equations.

The functions ϕ_i, ψ and the statistical characteristics $q(a; \vartheta)$, $q(a, \beta; \vartheta)$ of the random variable $\eta(t, \vartheta)$ should be chosen so that the problem should be easily solvable for $\vartheta = 0$, and with ϑ varying between 0 and 1 the system (5.1) should continuously transform into the initial system (1.1), (1.3) (for $\vartheta = 1$).

Using the equations describing the change in the optimum functions $v^\circ(x, \xi, \eta; \vartheta)$, $\zeta^\circ[x, \xi, \eta; \vartheta]$ for the problem (5.1), (5.2) which occurs when ϑ is changed, one can find the optimum solution for the original problem. In order to obtain these equations, it is sufficient to formulate Equations (4.7) and (4.8) for the problem (5.1), (5.2) and differentiate these equations with respect to ϑ. Although the resulting equations are also rather complex, one can use the intial solutions v° and ζ° at $\vartheta = 0$ to evolve a procedure for an approximate numerical solution. This approach is also convenient because by continuously varying the problem through varying ϑ, we shall be concerned with the branch of the solution $v^\circ(x, \xi, \eta; \vartheta)$, $\zeta^\circ[x, \xi, \eta; \vartheta]$ which, starting with a stable initial solution $v^\circ(x, \xi, \eta; 0)$, $\zeta^\circ[x, \xi, \eta; 0]$ at $\vartheta = 0$, will give for $\vartheta > 0$ a stable optimum system. (The function $v^\circ(x, \xi, \eta; \vartheta)$ for $\vartheta > 0$ will be found to be positive-definite.) The continuous deformation of the system is also convenient in investigating the existence of a solution for the optimum problem*.

6. The choice of parameters of a linear system which will minimize the quadratic quality criterion. The aim of the present section is to give an illustration of the approach summarized in

* This is illustrated below in the case of an example involving the minimization of the quadratic functional in the linear system (see Sections 6 and 7).

Section 4. Let us consider a system* which can be described by the linear equations (1.1)

$$\frac{dx_i}{dt} = \sum_{j=1}^{2} a_{ij}(\eta)\, x_j + m_i \xi \qquad (j = 1, 2) \tag{6.1}$$

and we shall seek the controlling action in the form

$$\dot{\xi} = \zeta\,[x_1,\, x_2,\, \xi,\, \eta(t)] \tag{6.2}$$

using the minimizing condition for the quadratic functional (2.3)

$$J_\zeta\,[x_{10},\ x_{20},\ \xi_0,\ \eta_0] =$$

$$= \int_0^\infty M\,\{[x_1^2(t) + x_2^2(t) + \xi^2(t) + \dot{\xi}^2(t)]\,/\,x_{10},\ x_{20},\ \xi_0,\ \eta_0,\ t_0 = 0\}\, dt = \min.$$

We shall confine our attention to the case where the variable $\eta(t)$ assumes only two values, namely, $\eta = \eta_1$ and $\eta = \eta_2$, with transition probabilities given by

$$P\,[\eta_i \to \eta_j \text{ during the time } \Delta t] = p_{ij}\,\Delta t + o(\Delta t) \qquad (i \neq j) \tag{6.3}$$

The functions $q(a)$ and $q(a,\ \beta)$ (p. 628) will be of the form

$$\begin{gathered} q(\eta_1) = p_{12}, \qquad q(\eta_2) = p_{21} \\ q(\eta_1,\ \beta) = q(\eta_1) = p_{12} \quad \text{for } \beta = \eta_2 \\ q(\eta_2,\ \beta) = q(\eta_2) = p_{21} \quad \text{for } \beta = \eta_1 \end{gathered}$$

The choice of the parameters for a linear system ensuring good quality has been discussed by Chetaev [2] .

The minimization of the quadratic quality criterion was formulated and investigated by Krasovskii and Fel'dbaum (see [14] and the bibliography therein).

The Liapunov-function method has been applied to the optimisation of control systems by Bertram and Kalman [15] .

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* The analysis given in the present section for the set of two equations (6.1) can be extended to an nth-order system; however the calculations become very involved.

Let us write down Equations (4.7), (4.8) for the problem defined by (6.3) in the form

$$\sum_{i=1}^{2} \frac{\partial v^\circ (x_1, x_2, \xi, \eta_l)}{\partial x_i} \left[\sum_{j=1}^{2} a_{ij}(\eta_l) x_j + m_i \xi \right] + \frac{\partial v^\circ (x_1, x_2, \xi, \eta_l)}{\partial \xi} \zeta^\circ [x_1, x_2, \xi, \eta_l] +$$
$$+ p_{lk} [v^\circ (x_1, x_2, \xi, \eta_k) - v^\circ (x_1, x_2, \xi, \eta_l)] +$$
$$+ x_1^2 + x_2^2 + \xi^2 + (\zeta^\circ [x_1, x_2, \xi, \eta_l])^2 = 0 \quad (l = 1, 2;\ k \neq l) \qquad (6.4)$$

$$\frac{\partial v^\circ (x_1, x_2, \xi, \eta_l)}{\partial \xi} + 2\zeta^\circ [x_1, x_2, \xi, \eta_l] = 0 \quad (l = 1, 2) \qquad (6.5)$$

Solving Equation (6.5) for ζ° and substituting into (6.4), we obtain the following partial differential equations for the optimum Liapunov function:

$$\sum_{i=1}^{2} \frac{\partial v^\circ (x, \xi, \eta_l)}{\partial x_i} \left[\sum_{j=1}^{2} a_{ij} (\eta_l) x_j + m_i \xi \right] + p_{lk} [v^\circ (x, \xi, \eta_k) - v^\circ (x, \xi, \eta_l)] -$$
$$- \frac{1}{4} \left[\frac{\partial v^\circ (x, \xi, \eta_l)}{\partial \xi} \right]^2 = - x^1{}_2 - x_2^2 - \xi^2 \quad (l = 1,2;\ k \neq l) \qquad (6.6)$$

The solution of these equations should be sought in the following quadratic form (see [5, Chapt. IV]):

$$v^\circ (x_1, x_2, \xi, \eta_l) = \sum_{i, j=1}^{2} [b_{ij} (\eta_l) x_i x_j + b_i (\eta_l) x_i \xi] + c (\eta_l) \xi^2 \quad (l = 1, 2) \qquad (6.7)$$

Substituting the right-hand side of (6.7) into (6.6) and equating the coefficients of equal powers of x_i and ξ, we obtain a system of quadratic equations for the coefficients b_{ij}, b_i, c.

Having found the solution of this system of equations, for which the forms given by (6.7) are positive-definite, in accordance with the results of Section 4, the optimum function can be determined from Equation (6.5) in the required form

$$\zeta^\circ = - \frac{1}{2} [b_1 (\eta_l) x_1 + b_2 (\eta_l) x_2 + 2c (\eta_l) \xi] \qquad (l = 1, 2) \qquad (6.8)$$

7. The existence of the optimum control law for the problem (6.3). We shall now discuss the existence of the solution v° for the partial differential equation (6.6). The system of Equations (6.1) cannot always be stabilized by choosing the control law (6.2) with given a_{ij} and m_i. Moreover, the solutions of Equations (6.6) which are positive-definite functions $v^\circ(x, \xi, \eta)$ do not always exist either. We shall define a control law $\dot{\xi} = \zeta^{(q)}$ as admissible if with such a law the system (6.1) is asymptotically stable in probability and the integral (6.3) is finite for any initial condition. The existence of the admissible (and optimum) control in the deterministic case has been investigated by F.M. Kirillova. In the present paper we shall not consider the existence of the admissible solution for the stochastic system (6.1), since this will be discussed in a separate paper. We shall merely note that for a given control law $\dot{\xi} = \zeta^{(q)}$, in order that the system should be asymptotically stable in probability and in order that the integral (6.3) should remain finite, it is sufficient to have a positive-definite function v, having a positive-definite derivative $dM\{v\}/dt$, where v and $dM\{v\}/dt$ satisfy estimates [7, pp. 815-823] characteristic for quadratic forms.

The main aim of the present section is to show that an optimum control exists if an admissible control exists.

Let us set up the auxiliary set of equations*

$$\frac{dx_i}{dt} = \vartheta\left[\sum_{j=1}^{2} a_{ij}(\eta)\, x_j + m_i\, \xi\right] - (1-\vartheta)\, x_i \qquad (i = 1,\, 2;\ 0 \leqslant \vartheta \leqslant 1) \tag{7.1}$$

$$\dot{\xi} = \zeta\,[x_1,\ x_2,\ \xi,\ \eta;\ \vartheta] \tag{7.2}$$

where (7.2) is so chosen that for each $\vartheta \in [0, 1]$ the system is asymptotically stable (in probability) and

$$J_\zeta\,[x_{10},\ x_{20},\ \xi_0,\ \eta_0,\ \vartheta] \tag{7.3}$$
$$= \int_0^\infty M\,\{[x_1^2(t) + x_2^2(t) + \xi^2(t) + \dot{\xi}^2(t)]\,/\,x_{10},\ x_{20},\ \xi_0,\ \eta_0,\ t_0 = 0\}\, dt = \min$$

When $\vartheta = 0$, the problem can be solved at once since the basic equations are

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* In the present case, the existence of the optimum solution can be established more simply and by direct means; however we shall give the general analysis which will apply to all cases and which illustrates the approach to the problem described above in Section 5.

$$-\sum_{j=1}^{2}\frac{\partial v^\circ(x,\xi,\eta)}{\partial x_i}x_i+\frac{\partial v^\circ(x,\xi,\eta)}{\partial \xi}\zeta^\circ[x,\xi,\eta]=-x_1^2-x_2^2-\xi^2-(\zeta^\circ[x,\xi,\eta])^2$$

$$2\zeta^\circ[x,\xi,\eta]=-\frac{\partial v^\circ(x,\xi,\eta)}{\partial \xi} \tag{7.4}$$

and can be satisfied by choosing

$$v^\circ=b_{11}x_1^2+b_{22}x_2^2+c\xi^2 \tag{7.5}$$

where the coefficients b_{ii} and c can be determined from the equations $2b_{ii}=1$, $c^2=1$ which were obtained by substituting (7.5) into (7.4). These equations have the positive solutions

$$b_{11}=\frac{1}{2},\quad b_{22}=\frac{1}{2},\quad c=1 \tag{7.6}$$

If for some $\vartheta>0$ the problem of the existence of an optimum Liapunov function v° can be solved, then the solution (a positive-definite function v°) satisfies the equations (6.6), i.e.

$$\sum_{i=1}^{2}\frac{\partial v^\circ}{\partial x_i}\left[\vartheta\sum_{j=1}^{2}a_{ij}x_j-(1-\vartheta)x_i+m_i\xi\vartheta\right]-\frac{1}{4}\left[\frac{\partial v^\circ}{\partial \xi}\right]^2+$$

$$+p_{lk}[v^\circ(x,\xi,\eta_k;\vartheta)-v^\circ(x,\xi,\eta_l;\vartheta)]=-x_1^2-x_2^2-\xi^2\qquad(l=1,2;\ k\neq l)$$

Let us consider the change in the solution v° of these equations with varying ϑ. Differentiating these equations with respect to ϑ and denoting $\partial v^\circ/\partial\vartheta$ by $\alpha[x,\xi,\eta;\vartheta]$, we have

$$\sum_{i=1}^{2}\frac{\partial \alpha}{\partial x_i}\left[\vartheta\sum_{j=1}^{2}a_{ij}x_j+\vartheta m_i\xi-(1-\vartheta)x_i\right]-\frac{1}{2}\frac{\partial v^\circ}{\partial \xi}\frac{\partial \alpha}{\partial \xi}+ \tag{7.7}$$

$$+p_{lk}[\alpha(x,\xi,\eta_k;\vartheta)-\alpha(x,\xi,\eta_l;\vartheta)]=F_l(x,\xi;\vartheta)\qquad(l=1,2;\ k\neq l)$$

where F_l is a quadratic form in x, x_2, ξ. If the problem has a solution for $\vartheta^*\geqslant 0$, then in view of the results given in [7] (pp. 815-820), the set of equations (7.1) will be asymptotically stable in the mean square. However, in that case (7.7) will have a solution for $\vartheta=\vartheta^*$ and this solution will have the quadratic form $\alpha(x_1,x_2,\xi,\eta;\vartheta)$. In fact, these equations can be written down in the form

$$\left(\frac{dM\{\alpha\}}{dt}\right)_{(7.1)(7.2)}=F_l(x,\xi;\vartheta) \tag{7.8}$$

where the symbol $(dM\{\alpha\}/dt_{(7.1)(7.2)}$ denotes the derivative of α

subject to the optimum system (7.1), (7.2) at $\vartheta = \vartheta^*$.

It also follows from [7] that under our conditions the solution a of (7.8) will exist. By equating coefficients of equal powers of x_i and ξ in (7.7), we obtain a set of differential equations for the functions $b_{ij}(\vartheta, \eta)$, $b_i(\vartheta, \eta)$, $c(\vartheta, \eta)$ (p. 639). It follows from the above discussion that the resulting set of equations can be solved for $db_{ij}/d\vartheta$, $db_i/d\vartheta$, $dc/d\vartheta$ which are the coefficients of the function a. Finally, in order to establish the existence of the solution of the optimum problem (6.3), i.e. the problem (7.4) for $\vartheta = 1$, it is sufficient to verify that these differential equations for b_{ii}, b_i, c can be integrated with respect to ϑ over the interval [0, 1] subject to the initial conditions (7.6). However, one can see that this integration would be impossible only if (7.7) could not be solved on approaching $\vartheta = \vartheta^*$ or if $b_{ij} \to \infty$, $b_i \to \infty$, $c \to \infty$ for $\vartheta \to \vartheta - 0$. However, these difficulties are only possible for those values of ϑ^* for which the system (7.1), (7.2) either loses its asymptotic stability on the average or $v^\circ(x, \xi, \eta; \vartheta)$ increases without limit at finite points x, ξ for $\vartheta \to \vartheta - 0$. In the first case, the function v° will no longer be positive-definite for $\vartheta \to \vartheta^* - 0$. However, these changes of v° are excluded, since for $\vartheta < \vartheta^*$ the function v° is equal to the optimum value of the functional J_ζ which, at the points x, ξ lying on a circle of unit radius, is uniformly bounded over $\vartheta \in [0, 1]$ both from below (due to the boundedness of the coefficients of the system in (7.1), (7.2)) and from above (since, in our view, for the system (6.1), (6.2) and consequently also for the system (7.1), (7.2)) a permissible control $\zeta^{(q)}$ exists for which the value of the functional I_ζ is not less than I_ζ°, i.e. $v^\circ \leqslant I_{\zeta(q)}$, this permissible control is obtained for the system (7.1), (7.2) if we assume $\zeta^{(q)} = \vartheta\,\zeta - (1 - \vartheta)\xi$, where ζ is the permissible control for the initial problem.

It follows from the above discussion that it is possible to extend the solution of (7.6) right up to $\vartheta = 1$, i.e. it follows that the optimum solution of the problem (6.3) does, in fact, exist.

In conclusion, we note that the numerical integration of the differential equations for b_{ij}, b_i, c on $v \in [0, 1]$ can be used to determine the approximate value of the optimum Liapunov function v°.

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